Klein-Gordon and Dirac particles in non-constant scalar-curvature background

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Abstract

The Klein-Gordon and Dirac equations are considered in a semi-infinite lab ($x > 0$) in the presence of background metrics $ds^2 = u^2(x)\eta_{\mu\nu}dx^\mu dx^\nu$ and $ds^2 = -dt^2 + u^2(x)\eta_{ij}dx^i dx^j$ with $u(x) = e^{\pm gx}$. These metrics have non-constant scalar-curvatures. Various aspects of the solutions are studied. For the first metric with $u(x) = e^{gx}$, it is shown that the spectrums are discrete, with the ground state energy $E^2_{min} = p^2c^2 + g^2c^2\hbar^2$ for spin-0 particles. For $u(x) = e^{-gx}$, the spectrums are found to be continuous. For the second metric with $u(x) = e^{-gx}$, each particle, depends on its transverse-momentum, can have continuous or discrete spectrum. For Klein-Gordon particles, this threshold transverse-momentum is $\sqrt{3}g/2$, while for Dirac particles it is $g/2$. There is no solution for $u(x) = e^{gx}$ case. Some geometrical properties of these metrics are also discussed.

1 Introduction

Studying the quantum mechanical effects of gravity is an important and interesting branch of physics which has been started from the early days of quantum mechanics. The simplest example of these effects is the behavior of the nonrelativistic spinless quantum particle, i.e. the Schrodinger equation, in the presence of constant gravity [1]. This phenomenon has been experimentally verified by the famous experiment of Collela et al. [2]. The latest of these experiments is one reported by Nesvizhevsky et al., in which the quantum energy levels of neutrons in the Earth's gravitational field have been measured [3, 4]. Other aspects of gravitational effects in quantum physics are appeared, for example, in neutrino oscillation in gravitational background [5, 6, 7], Berry phase of spin-1/2 particles moving in a space-time with torsion [8, 9], etc.

Another branch of researches in this area is the study of the behaviors of Dirac and Klein-Gordon particles in the curved background and distinguishes their physical characteristics. This is an interesting subject since it makes clear the importance of the spin of the particles, which is a purely quantum mechanical property, in the gravitational interactions. Chandrasekhar, for example, has considered the Dirac equation in a Kerr-geometry background [10], with results which have been followed by others [11, 12].

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A quick review in the literature of this field shows that the number of integrable models is very few. For example in the case of Schwarzschild metric, where the metric's components depend only on one spatial coordinate r, the problem is too complicated to be solved analytically. So trying to exactly solve some relativistic quantum mechanical examples in curved background, may shed light on this important topic and can help us to achieve more insight into the realistic problems.

A class of background metrics which can be considered in this area, is one which depends only on one spatial variable x. In [13], a semi-infinite laboratory ($x > 0$) has been considered in background metric

$$ds^2 = u^2(x)(-dt^2 + dx^2) + dy^2 + dz^2. \tag{1}$$

In fact, an infinite barrier has been assumed in $x < 0$ region. The Klein-Gordon and Dirac equations have been studied for the constant-gravity approximation of this background, i.e. $u(x) \simeq 1 + gx$, and some interesting features of this problem have been discussed. For example it has been shown that in the case of zero transverse-momentum, there exists an exact relation between the squares of energy eigenvalues of spin-1/2 and spin-0 particles with same masses: $E^2_{\text{Dirac}} = E^2_{\text{KG}} + mg\hbar c$. So the gravity clearly distinguishes between the Fermions and Bosons. The scalar-curvature of metric (1) is $R = 0$.

In [14], another member of this class, that is the metric

$$ds^2 = -dt^2 + dx^2 + u^2(x)(dy^2 + dz^2), \tag{2}$$

has been considered. For the case $u(x) = e^{-gx}$, the Klein-Gordon and Dirac equations have been studied and their eigenfunctions have been obtained. As an exact result, it has been shown that the spin-0 particles have specific ground state energy : $E_{\text{KG}} \geq \sqrt{m^2c^4 + g^2c^2\hbar^2}$, while the spin-1/2 particles have the natural rest-mass energy ground state : $E_{\text{Dirac}} \geq mc^2$. The scalar curvature of metric (2) is $R = 6g^2$. So both metrics (1) and (2) have constant scalar curvatures.

In this paper, we are going to study the behaviors of relativistic spin-0 and spin-1/2 particles in one-variable-dependent-metrics with non-constant scalar-curvature. For a semi-infinite lab with an infinite potential barrier in $x < 0$, we consider the following two metrics:

$$ds^2 = u^2(x)(-dt^2 + dx^2 + dy^2 + dz^2), \tag{3}$$

$$ds^2 = -dt^2 + u^2(x)(dx^2 + dy^2 + dz^2). \tag{4}$$

It is worth nothing that in metrics (1) and (2), only two of the metrics' components are nontrivial, while here we study the next steps and take three (in eq.(4)) and four (in eq.(3)) of the components nontrivial.

To make our problems more solvable, we assume $u(x) = e^{\pm gx}$, which are the similar choices that have been considered in the preceding cases. As the result, both metrics (3) and (4) gain the x-dependent scalar curvatures, and the Dirac and Klein-Gordon equations show interesting properties with significant different behaviors. It is worth mentioning that changing the variable from x to X, defined by $X = \int u(x)dx$, transforms the metric (4) to (2), but with new $u(x)$, i.e. $u(x) \to f(X)$. Now if one decides to study both metrics (2)

and (4) with the same $u(x)$, as we do in this paper, then these two metrics are independent.

The plan of the paper is as follows: In section 2, after fixing our notations, we discuss the Klein-Gordon and Dirac equations in background metric (3) with $u(x) = e^{\pm gx}$. It is shown that the spectrum of energy eigenvalues are discrete for $u(x) = e^{gx}$ and continuous for $u(x) = e^{-gx}$. The discrete spectrums are compared numerically and the geometrical properties of the metrics, including their geodesics, are discussed. The importance of geodesic in this problem is that it determines how much it is possible to consider the $x = 0$-plane as the floor of the laboratory. Noting that the endpoints of all classical falling particles are the floor, then only if the classical trajectories finally intersect $x = 0$, this plane can be considered as floor, otherwise not. We see that this is the case for $u(x) = e^{-gx}$.

In section 3, the same is done for metric (4) and it is shown that only the case $u(x) = e^{-gx}$ is consistent with the desired boundary conditions. For this u, it is shown that in both cases, i.e. spin-0 and spin-1/2 particles, the spectrums have interesting properties. They are continuous for $p < p_0$ and discrete for $p > p_0$. p is the transverse-momentum of the particles and the value of p_0 is different for Dirac and Klein-Gordon particles. The geometrical properties of metric (4) are also discussed. Finally in section 4, we review our main results and bring some comments on metric

$$ds^2 = u^2(x)(-dt^2 + dx^2) + v^2(x)(dy^2 + dz^2), \tag{5}$$

which is somehow a combination of two metrics (1) and (2).

2 Conformally-flat metric $ds^2 = u^2(x)\eta_{\mu\nu}dx^\mu dx^\nu$

In a space-time with metric $g_{\mu\nu}$, the Klein-Gordon equation in $c = \hbar = 1$ unit is

$$\left[\frac{1}{\sqrt{-\det g_{\mu\nu}}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-\det g_{\mu\nu}}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right) - m^2\right]\psi_{\mathrm{KG}} = 0. \tag{6}$$

The Dirac equation in curved background is

$$\left[\gamma^a\left(\partial_a + \Gamma_a\right) - m\right]\psi_{\mathrm{D}} = 0. \tag{7}$$

γ^as are the Dirac matrices and Γ_as are spin connections which can be obtained from tetrads e^a through

$$\begin{aligned} de^a + \Gamma^a{}_b \wedge e^b =&0, \\ \Gamma^a{}_b =&\Gamma^a{}_{cb}e^c, \\ \Gamma_a =&-\frac{1}{8}\left[\gamma_b, \gamma_c\right]\Gamma^c{}_a{}^b. \end{aligned} \tag{8}$$

The boundary conditions of the semi-infinite lab ($x > 0$) with an infinite potential barrier at $x = 0$ are as follows. For Schrodinger equation, the boundary condition is $\lim_{x\to 0}\psi_{\mathrm{sch}} = 0$, which comes from the fact that the Schrodinger equation is second order in x, so ψ_{sch} must be continuous at $x = 0$. The same is true for Klein-Gordon equation, so the same boundary condition exists:

$$\psi_{\mathrm{KG}}(0) = 0. \tag{9}$$

But the Dirac equation is of first order in x and therefore ψ_{D} can be discontinuous at $x=0$, if the potential goes to infinity there. In this case, it can be shown that the desired boundary condition is [13]

$$\left(\gamma^1-1\right)\psi_{\mathrm{D}}(0)=0. \tag{10}$$

The square-integrability of wavefunctions ψ_{KG} and ψ_{D} in $(0,\infty)$ region also leads to

$$\lim_{x\to\infty}\sqrt{\det g_{\mu\nu}}\,|\psi|^2\to 0. \tag{11}$$

2.1 The Klein-Gordon equation

For the metric (3), the Klein-Gordon equation (6) becomes

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+2\frac{u'}{u}\frac{\partial}{\partial x}-m^2u^2\right)\psi_{\mathrm{KG}}=0. \tag{12}$$

Since u depends only on x, one may seek a solution whose functional form is $\psi_{\mathrm{KG}}\left(t,x,y,z\right)=\exp\left(-iEt+ip_2y+ip_3z\right)\psi_{\mathrm{KG}}(x)$. Then ψ_{KG} satisfies

$$\left(E^2-p_2^2-p_3^2+\frac{d^2}{dx^2}+2\frac{u'}{u}\frac{d}{dx}-m^2u^2\right)\psi_{\mathrm{KG}}(x)=0. \tag{13}$$

The above equation becomes solvable if we assume u'/u=constant. So we take

$$u(x)=e^{\pm gx}. \tag{14}$$

Let us first consider $u(x)=e^{gx}$ case.

Substitute $u(x)=e^{gx}$ into eq.(13), and defining $\phi(x)$ through

$$\psi_{\mathrm{KG}}=e^{-gx}\phi(x), \tag{15}$$

one finds

$$\frac{d^2\phi\left(x\right)}{dx^2}+\left[g^2\left(\lambda^2-1\right)-m^2e^{2gx}\right]\phi\left(x\right)=0, \tag{16}$$

in which

$$\begin{aligned} p^2 :=& p_2^2+p_3^2, \\ \lambda^2 :=& \frac{E^2-p^2}{g^2}. \end{aligned} \tag{17}$$

In terms of new variable $X=(m/g)e^{gx}$, eq.(16) reduces to the modified Bessel equation

$$X^2\frac{d^2\phi\left(X\right)}{dX^2}+X\frac{d\phi\left(X\right)}{dX}-\left(X^2+1-\lambda^2\right)\phi\left(X\right)=0 \tag{18}$$

with solutions $K_\nu(X)$ and $I_\nu(X)$ ($\nu=\sqrt{1-\lambda^2}$). Using the asymptotic behaviors of Bessel functions

$$I_\nu(z)\sim z^\nu \quad , \quad K_\nu(z)\sim\frac{1}{z^\nu}\ (\nu\neq 0), \tag{19}$$

in the limit $z\to 0$ (with similar relations for Bessel functions $J_\nu(z)$ and $Y_\nu(z)$, respectively), and

$$I_\nu(z)\sim\frac{e^z}{\sqrt{z}} \quad , \quad K_\nu(z)\sim\frac{e^{-z}}{\sqrt{z}}, \tag{20}$$

in the limit $z \to \infty$, it can be easily seen that the boundary condition (11) discards $I_\nu(X)$:

$$\lim_{x\to\infty} \sqrt{\det g_{\mu\nu}}\, |\psi_{\rm KG}|^2 \sim e^{2(\nu+1)gx} \to \infty. \tag{21}$$

So the wavefunction becomes

$$\psi_{\rm KG} = Ce^{-gx} K_\nu(\frac{m}{g} e^{gx}). \tag{22}$$

The boundary condition at $x = 0$, eq.(9), forces us to take ν a pure imaginary number, since K_ν becomes zero only when ν is pure imaginary [15]. So one has to take $\lambda^2 > 1$, which results the ground state energy of the spin-0 particles in background metric (3) with $u(x) = e^{gx}$ to be:

$$E^2 \geq E_{min}^2 = p^2c^2 + g^2c^2\hbar^2. \tag{23}$$

Other energy eigenvalues can be obtained by equation:

$$K_{i\sqrt{\lambda^2-1}}(\frac{m}{g}) = 0, \tag{24}$$

which clearly results in a discrete spectrum.

In $u(x) = e^{-gx}$ case, a similar argument leads to

$$\psi_{\rm KG} = e^{gx}\left[C_1 I_\nu(\frac{m}{g}e^{-gx}) + C_2 K_\nu(\frac{m}{g}e^{-gx})\right], \tag{25}$$

with $\nu = \sqrt{1-\lambda^2}$ and λ defined through eq.(17). Since $\sqrt{\det g_{\mu\nu}} = e^{-4gx}$, the function $I_\nu(\frac{m}{g}e^{-gx})$ satisfies the boundary condition (11) for all positive νs. But one has $\sqrt{\det g_{\mu\nu}}\, |\psi_{\rm KG}|^2 \sim e^{2(\nu-1)gx}$ for $K_\nu(\frac{m}{g}e^{-gx})$, which satisfies (11) only if $\Re(\nu) < 1$. For $\lambda^2 < 1$, $\nu = \sqrt{1-\lambda^2} < 1$ is a real number and for $\lambda^2 > 1$, ν is pure-imaginary number (so $\Re(\nu) = 0$). Therefore in all cases, one has $\Re(\nu) < 1$ which implies that C_1 and C_2 in eq.(25) are arbitrary non-zero constants, up to the normalization condition of wavefunction. The boundary condition (9) then gives

$$C_1 I_\nu(\frac{m}{g}) + C_2 K_\nu(\frac{m}{g}) = 0, \tag{26}$$

which results in a continuous energy spectrum.

2.2 The Dirac equation

For metric (3), the nonvanishing $\Gamma^a{}_b$s are $\Gamma^0{}_1 = \Gamma^1{}_0 = (u'/u^2)e^0$, $\Gamma^2{}_1 = -\Gamma^1{}_2 = (u'/u^2)e^2$ and $\Gamma^3{}_1 = -\Gamma^1{}_3 = (u'/u^2)e^3$, from which $\Gamma^0{}_0{}^1 = -\Gamma^1{}_0{}^0 = \Gamma^2{}_2{}^1 = -\Gamma^1{}_2{}^2 = \Gamma^3{}_3{}^1 = -\Gamma^1{}_3{}^3 = u'/u^2$. Therefore one finds $\Gamma_1 = 0$ and $\Gamma_a = -(u'/2u^2)\gamma_1\gamma_a$ $(a = 0, 2, 3)$. Noting that $\gamma^a\partial_a = \gamma^a e^\mu_a \partial_\mu$, in which $e^\mu_a = \mathrm{diag}(u^{-1}, u^{-1}, u^{-1}, u^{-1})$, the Dirac equation (7) leads to:

$$\left[\frac{1}{u}(\gamma^0\frac{\partial}{\partial t} + \gamma^k\frac{\partial}{\partial x^k}) + \frac{3}{2}\frac{u'}{u^2}\gamma^1 - m\right]\psi_{\rm D} = 0. \tag{27}$$

Since $u = u(x)$, it is natural to take $\psi_{\rm D}(t, x, y, z) = \exp(-iEt + ip_2y + ip_3z)\psi_{\rm D}(x)$. Defining $\tilde{\psi}_{\rm D}(x)$ through

$$\psi_{\rm D}(x) = u^{-3/2}\tilde{\psi}_{\rm D}(x), \tag{28}$$

eq.(27) then results in

$$(O_1 + O_2)\phi_{\rm D}(x) = 0. \tag{29}$$

In above equation, O_1, O_2 and $\phi_{\rm D}(x)$ are:

$$O_1 = -\,iE\gamma^1 + \gamma^0 \frac{d}{dx} - mu\gamma^1\gamma^0, \tag{30}$$
$$O_2 = i(p_2\gamma^2 + p_3\gamma^3)\gamma^1\gamma^0, \tag{31}$$
$$\phi_{\rm D}(x) = \gamma^1\gamma^0\tilde{\psi}_{\rm D}(x). \tag{32}$$

Using the fact that $[O_1, O_2] = 0$, it may be possible to choose the common eigenspinors for O_1 and O_2. The eigenvalues of O_2 are $\pm ip$, with p defined in eq.(17) and each eigenvalues are two-fold degenerate. For ip-eigenvalue, the eigenspinors are

$$\chi_1 = \begin{pmatrix} i(p_2 - p)/p_3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \chi_2 = \begin{pmatrix} 0 \\ 0 \\ i(p_2 + p)/p_3 \\ 1 \end{pmatrix}. \tag{33}$$

So one can choose $\phi_{\rm D}(x) = \phi_1'(x)\chi_1 + \phi_2(x)\chi_2$ and determines the unknown functions $\phi_1'(x)$ and $\phi_2(x)$ such that $\phi_{\rm D}(x)$ satisfies (29). If we write $\phi_{\rm D}$ as

$$\phi_{\rm D} = \begin{pmatrix} \phi_1 \\ \phi_1' \\ \phi_2' \\ \phi_2 \end{pmatrix}, \tag{34}$$

then ϕ_1 and ϕ_2' relate to ϕ_1' and ϕ_2 as following

$$\phi_1 = \frac{i(p_2 - p)}{p_3}\phi_1',$$
$$\phi_2' = \frac{i(p_2 + p)}{p_3}\phi_2. \tag{35}$$

Now it is sufficient to obtain two functions $\phi_1(x)$ and $\phi_2(x)$, from which $\phi_{\rm D}(x)$ and therefore $\psi_{\rm D}(x)$ will be determined. Noting that $O_1\phi_{\rm D} = -ip\phi_{\rm D}$, eqs.(30) and (34) result in:

$$\frac{d\phi_1}{dx} = p\phi_1 - (E + mu)\phi_2,$$
$$\frac{d\phi_2}{dx} = -p\phi_2 + (E - mu)\phi_1. \tag{36}$$

It can be also easily shown that the boundary condition (10) for $\psi_{\rm D}(0)$ reduces to the following boundary condition on ϕ_1 and ϕ_2:

$$(\phi_1(x) + \phi_2(x))\,|_{x=0} = 0. \tag{37}$$

Defining ψ_1 and ψ_2 through:

$$\psi_1 = \phi_1 + \phi_2,$$
$$\psi_2 = \phi_1 - \phi_2, \tag{38}$$

the differential equations (36) then lead to:

$$\frac{d\psi_1}{dx} = (p+E)\,\psi_2 - mu\psi_1,$$
$$\frac{d\psi_2}{dx} = (p-E)\,\psi_1 + mu\psi_2. \tag{39}$$

We first consider $u(x) = e^{gx}$ case. Introducing the new variable $X = (2m/g)e^{gx}$, the differential equation of ψ_1 becomes

$$X^2\frac{d^2\psi_1}{dX^2} + X\frac{d\psi_1}{dX} + \left(\lambda^2 + \frac{1}{2}X - \frac{1}{4}X^2\right)\psi_1 = 0, \tag{40}$$

where λ is defined in eq.(17). Defining $\tilde{\psi}_1$ through

$$\psi_1 = X^{-1/2}\tilde{\psi}_1, \tag{41}$$

eq.(40) leads to:

$$\frac{d^2\tilde{\psi}_1}{dX^2} + \left(-\frac{1}{4} + \frac{1/2}{X} + \frac{1/4+\lambda^2}{X^2}\right)\tilde{\psi}_1 = 0, \tag{42}$$

which is Whittaker differential equation with solution

$$\tilde{\psi}_1 = e^{-X/2}X^{i\lambda+1/2}\left[C_1 M\left(i\lambda, 1+2i\lambda, X\right) + C_2 U\left(i\lambda, 1+2i\lambda, X\right)\right]. \tag{43}$$

$M\left(a,c,x\right)$ and $U\left(a,c,x\right)$ are confluent hypergeometric functions. The boundary condition (11) implies $\psi_1(x\to\infty) = X^{-1/2}\tilde{\psi}_1(X)\mid_{X\to\infty}= 0$. But the asymptotic behavior of $M\left(a,c,x\right)$ is e^x/x^{c-a}, so $C_1 = 0$. The second boundary condition (37) results in $\psi_1(x=0) = X^{-1/2}\tilde{\psi}_1(X)\mid_{X=(2m/g)}= 0$, which leads to:

$$\left(\frac{2m}{g}\right)^{i\lambda} U(i\lambda, 1+2i\lambda, \frac{2m}{g}) = 0. \tag{44}$$

This equation determines the discrete energy eigenvalues of Dirac particles when they are in background metric (3) with $u(x) = e^{gx}$.

For $u(x) = e^{-gx}$, the same procedure leads to:

$$\psi_1 = e^{-Y/2}Y^{i\lambda}\left[C_1 M\left(1+i\lambda, 1+2i\lambda, Y\right) + C_2 U\left(1+i\lambda, 1+2i\lambda, Y\right)\right], \tag{45}$$

where λ is defined in eq.(17) and $Y \equiv (2m/g)e^{-gx}$. Here the boundary condition (11) implies $\psi_1(Y\to 0)\to 0$ which can not discard none of the constants C_1 and C_2. The energy eigenvalues can be obtained by (37), which results in

$$C_1 M\left(1+i\lambda, 1+2i\lambda, 2\frac{m}{g}\right) + C_2 U\left(1+i\lambda, 1+2i\lambda, 2\frac{m}{g}\right) = 0. \tag{46}$$

Since C_1 and C_2 are arbitrary constants, up to the normalization condition, eq.(46) results in a continuous energy spectrum.

It may be worth noting that wavefunctions of scalar and spin-1/2 particles in two background metrics which relate by a conformal transformation, can be obtained from each other if the particles are massless. If we consider the

conformal transformation $g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x)$, then if $R = 0$, one can show that $\Box\phi = 0 \to \bar{\Box}\bar{\phi} = 0$ in which [16]

$$\bar{\phi}(x) = \Omega(x)^{(2-n)/2}\phi(x). \tag{47}$$

n is the dimension of space-time. For massless fermions, one also has

$$\psi(x) \to \bar{\psi}(x) = \Omega(x)^{(1-n)/2}\psi(x). \tag{48}$$

In our problem, R is zero (for flat metric $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$). Now if we take $m = 0$ in eq.(16), the eqs.(15) and (16) result in

$$\psi_{\rm KG}(x) = u^{-1}\psi_{\rm KG}^{\rm flat}(x), \tag{49}$$

which is consistent with eq.(47). Also if we put $m = 0$ in eq.(27) and inserting eq.(28) into eq.(27), we obtain

$$\gamma^\mu \frac{\partial}{\partial x^\mu}\tilde{\psi}_{\rm D}\,(x) = 0, \tag{50}$$

which is the free Dirac equation in flat space-time, i.e. $\tilde{\psi}_{\rm D} = \psi_{\rm D}^{\rm flat}$. This shows $\psi_{\rm D}^{\rm non-flat} = u^{-3/2}\psi_{\rm D}^{\rm flat}$ which is again consistent with eq.(48).

2.3 Comparing the spectrums

For $u(x) = e^{gx}$, the spectrums of spin-0 and spin-1/2 particles can be obtained by eqs.(24) and (44), respectively. None of these equations can be solved analytically and only in $mc/g\hbar >> 1$ limit, an approximate solution can be obtained for eq.(24) [15]. The main difference between two spectrums is that the Klein-Gordon eigenvalues has a ground state, i.e. $\lambda_{\rm KG}^2 > 1$, but for Dirac particle both $\lambda_{\rm Dirac}^2 < 1$ and $\lambda_{\rm Dirac}^2 > 1$ cases are possible. To obtain the numerical values of energies (λs), one must fix the values of m and g and then finds the roots of two equations (24) and (44). For example for $mc/g\hbar = 0.1$, Table 1 shows some of the lowest energy levels of Dirac and Klein-Gordon particles. The energies can be found by using eq.(17): $E = \sqrt{p^2c^2 + (\lambda g c\hbar)^2}$.

Table 1: The lowest ten values of $\lambda_{\rm KG}$ and $\lambda_{\rm D}$ for $(mc/g\hbar) = 0.1$

$\lambda_{\rm KG}$	1.52	2.27	3.02	3.74	4.44	5.12	5.78	6.42	7.04	7.68
$\lambda_{\rm D}$	0.85	1.82	2.65	3.42	4.14	4.84	5.51	6.17	6.82	7.46

2.4 The metric properties

The scalar curvature of metric (3) is

$$R = 6u^{-3}u''. \tag{51}$$

So for $u = e^{gx}$, $R = 6g^2e^{-2gx}$ and for $u = e^{-gx}$ one has $R = 6g^2e^{2gx}$. Both Rs are x-dependent, and in $u = e^{-gx}$ case, the scalar-curvature R and the Kretschmann-invariant $K = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = 6g^4e^{4gx}$ diverge at $x \to \infty$.

The classical trajectories of particles in these backgrounds are also interesting. For $u = e^{\pm gx}$, one can show that

$$
\begin{aligned}
x(t) =& x(0) \mp \frac{1}{g} \ln\cosh\left[\sqrt{1-v_\perp^2}\, g\,(t-t_0)\right], \\
y(t) =& y(0) + v_{0y} t, \\
z(t) =& z(0) + v_{0z} t,
\end{aligned}
\tag{52}
$$

in which v_{0y} and v_{0z} are arbitrary constants and $v_\perp^2 = v_{0y}^2 + v_{0z}^2$. The x-component of velocity is

$$
v_x(t) = \mp\sqrt{1-v_\perp^2}\tanh\left[\sqrt{1-v_\perp^2}\, g\,(t-t_0)\right]. \tag{53}
$$

Note that in $\hbar = c \equiv 1$ unit, $v_x^2 + v_\perp^2 < 1$. Eq.(53) shows that for $u = e^{gx}$, $v_x(t)$ is always negative (the particles fall in $(-x)$-direction) and in $u = e^{-gx}$ case, $v_x(t) > 0$ and the particles fall in $(+x)$-direction (toward the singular region).

3 The $ds^2 = -dt^2 + u^2(x)\eta_{ij}dx^i dx^j$ metric

In this section we study the Klein-Gordon and Dirac particles in the presence of the metric (4):

$$
ds^2 = -dt^2 + u^2(x)\left(dx^2 + dy^2 + dz^2\right) \tag{4}
$$

.

3.1 The spin-0 particles

For metric (4), the Klein-Gordon equation (6) becomes

$$
\left[-\frac{\partial^2}{\partial t^2} + \frac{1}{u^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{u'}{u^3}\frac{\partial}{\partial x} - m^2\right]\psi_{\rm KG} = 0. \tag{54}
$$

To make the above equation solvable, we assume that $1/u^2 \propto u'/u^3$ or $u(x) = e^{\pm gx}$. Let us first consider $u(x) = e^{-gx}$ case.

Following the same steps of Sec.2.1, that is taking $\psi_{\rm KG}(t, \mathbf{r}) = \exp(-iEt + ip_2 y + ip_3 z)\psi_{\rm KG}(x)$, defining $\psi_{\rm KG}(x) = e^{gx}\phi(x)$ and $X = (k/g)e^{-gx}$, where

$$
k^2 = E^2 - m^2, \tag{55}
$$

eq.(54) then reduces to

$$
\left(X^2\frac{d^2}{dX^2} + X^2 - \eta_{\rm KG}^2\right)\phi(X) = 0, \tag{56}
$$

in which $\eta_{\rm KG} = p/g$ $(p^2 = p_2^2 + p_3^2)$. The solutions of the above differential equation are $\sqrt{X}J_\mu(X)$ and $\sqrt{X}Y_\mu(X)$. μ is defined through

$$
\mu = \sqrt{\eta^2 + \frac{1}{4}}. \tag{57}
$$

The limit $x \to \infty$ corresponds to $X \to 0$. Since $\sqrt{\det g_{\mu\nu}} = e^{-3gx}$ and $Y_\mu(X) \sim 1/X^\mu$ (for $\mu \neq 0$), the boundary condition (11) leads to $\sqrt{\det g_{\mu\nu}}\, |\psi_{\rm KG}|^2 \sim X^{2(1-\mu)}$. So for $\mu \leq \mu_0 = 1$, i.e. $p \leq p_0 = (\sqrt{3}/2)g$, both Bessel functions J_μ and Y_μ are acceptable and the Klein-Gordon wavefunction in background metric (4) with $u = e^{-gx}$ becomes

$$\psi_{\rm KG}\,(x) = e^{(1/2)gx} \left[C_1 J_\mu \left(\frac{k}{g} e^{-gx} \right) + C_2 Y_\mu \left(\frac{k}{g} e^{-gx} \right) \right], \qquad \left(p \leq \frac{\sqrt{3}}{2} g \right). \tag{58}$$

The eigenvalues are determined by eq.(9) which results in a continuous spectrum. For $\mu > 1$, $Y_\mu(X)$ does not satisfy the boundary condition (11). Therefore the wavefunction is

$$\psi_{\rm KG}\,(x) = C e^{(1/2)gx} J_\mu \left(\frac{k}{g} e^{-gx} \right), \qquad \left(p > \frac{\sqrt{3}}{2} g \right), \tag{59}$$

and its corresponding eigenvalues can be obtained by

$$J_\mu \left(\frac{k}{g} \right) = 0, \qquad \left(p > \frac{\sqrt{3}}{2} g \right). \tag{60}$$

If we consider $u = e^{gx}$, instead of eq.(56), one obtains

$$\left(Y^2 \frac{d^2}{dY^2} + Y^2 - \eta_{\rm KG}^2 \right) \phi\,(Y) = 0, \tag{61}$$

where $\psi_{\rm KG}\,(x) = e^{-gx} \phi\,(x)$ and $Y \equiv (k/g) e^{gx}$. Again the solutions are $\sqrt{Y} J_\mu\,(Y)$ and $\sqrt{Y} Y_\mu\,(Y)$, but now the $x \to \infty$ limit corresponds to $Y \to \infty$. Since the asymptotic behaviors of $J_\mu(Y)$ and $Y_\mu(Y)$ in the limit $Y \to \infty$ are:

$$\begin{aligned} J_\mu\,(Y) &\to \sqrt{\frac{2}{\pi Y}} \cos \left(Y - \mu \frac{\pi}{2} - \frac{\pi}{4} \right), \\ Y_\mu\,(Y) &\to \sqrt{\frac{2}{\pi Y}} \sin \left(Y - \mu \frac{\pi}{2} - \frac{\pi}{4} \right), \end{aligned} \tag{62}$$

and $\sqrt{\det g_{\mu\nu}} = e^{3gx} \sim Y^3$, the boundary condition (11) does not satisfy. So the Klein-Gordon equation has no solution in this case.

3.2 The spin-1/2 particles

For Dirac particles in the presence of metric (4), the procedure is similar to one introduced in Sec.2.2, but with some differences. Here, instead of eq.(28), one has $\psi_{\rm D}(x) = u^{-1} \tilde{\psi}_{\rm D}(x)$, and instead of (30), it is $O_1 = -iEu\gamma^1 + \gamma^0 d/dx - mu\gamma^1\gamma^0$, which again $[O_1, O_2] = 0$. Finally one arrives at, instead of eq.(36), the following equations for ϕ_1 and ϕ_2:

$$\begin{aligned} \frac{d\phi_1}{dx} &= p\phi_1 - (E+m)\,u\phi_2, \\ \frac{d\phi_2}{dx} &= -\,p\phi_2 + (E-m)\,u\phi_1. \end{aligned} \tag{63}$$

For $u = e^{-gx}$, by introducing $X = (k/g)e^{-gx}$, one finds

$$\left(X^2 \frac{d^2}{dX^2} + X^2 - \eta_{\mathrm{D}}^2\right) \phi_1 (X) = 0. \tag{64}$$

In this equation, $\eta_{\mathrm{D}} = \sqrt{p\,(p+g)}/g$, and k is defined through eq.(55). It is interestingly seen that we arrive at same equation for Klein-Gordon and Dirac particles, i.e. eqs.(56) and (64), but with different η. Like the pervious section, for $\mu \leq \mu_0 = 1$, $\left(\mu = \sqrt{\eta_{\mathrm{D}}^2 + 1/4}\right)$, which now indicates $p \leq p_0' = g/2$, both $\sqrt{X} J_\mu(X)$ and $\sqrt{X} Y_\mu(X)$ satisfy the boundary condition (11), and therefore:

$$\phi_1 = e^{-(1/2)gx} \left[C_1 J_\mu \left(\frac{k}{g} e^{-gx}\right) + C_2 Y_\mu \left(\frac{k}{g} e^{-gx}\right)\right], \qquad \left(p \leq \frac{g}{2}\right). \tag{65}$$

The second function, ϕ_2, can be obtained by the first equation of (63) and the resulting eigenvalues are continuous.

For $\mu > 1$, $Y_\mu(X)$ is not acceptable and therefore

$$\phi_1 = C e^{-(1/2)gx} J_\mu \left(\frac{k}{g} e^{-gx}\right), \qquad \left(p > \frac{g}{2}\right). \tag{66}$$

Using eqs.(66) and (63), the $x = 0$ boundary condition, i.e. eq.(37), then leads to

$$\left[1 + \frac{2p + g}{2\,(E+m)}\right] J_\mu \left(\frac{k}{g}\right) + \frac{k}{E+m} J'_\mu \left(\frac{k}{g}\right) = 0, \qquad \left(p > \frac{g}{2}\right), \tag{67}$$

where J'_μ is the derivative of J_μ with respect to its argument. The above equation determines the discrete eigenvalues of Dirac particle in the presence of metric (4) with $u = e^{-gx}$.

In the case $u = e^{gx}$, the result is same as in the Klein-Gordon case and the $x \to \infty$ boundary condition can not be satisfied.

3.3 Spin-0 and spin-1/2's spectrums comparison

The interesting feature of the spectrums of both equations, i.e. the Klein-Gordon and Dirac equations, is that they are continuous for low transverse-momentum ($p \leq p_0$) and discrete for higher one ($p > p_0$). The value of p_0, however, differs for the two cases, i.e. it is spin-dependent. $(p_0)_{\mathrm{KG}} = \left(\sqrt{3}/2\right) g$ and $(p_0)_{\mathrm{Dirac}} = g/2$. To find the energy eigenvalues, one must fix the m, g and p values in eqs.(60) and(67) and then solves these equations numerically. For example for $mc/g\hbar = 0.1$ and $p/(g\hbar) = 2$, the ten lowest k-values for Klein-Gordon and Dirac particles are shown in Table 2. The energy eigenvalues can be obtained from this table by using $E = \sqrt{k^2 + m^2 c^4}$.

3.4 Classical characteristics of metric

The scalar curvature of metric (4) is

$$R = 4u^{-3} u'' - 2u^{-4} u'^2. \tag{68}$$

Table 2: The lowest ten values of $k_{\rm KG}$ and $k_{\rm D}$ in $gc\hbar$ unit for $mc/g\hbar = 0.1$ and $p/(g\hbar) = 2$

$k_{\rm KG}$	0	5.21	8.50	11.71	14.88	18.05	21.21	24.36	27.51	30.66
$k_{\rm D}$	0	5.15	8.41	11.62	14.79	17.96	21.11	24.27	27.42	30.57

So for $u = e^{-gx}$, we have $R = 2g^2 e^{2gx}$. Besides R, the Kretschmann-invariant K also diverges at $x \to \infty$, $K = 2g^4 e^{4gx}$.

The equation of geodesics of this metric can be found as follows:

$$\begin{aligned} y =& k_1 z + k_2, \\ x\left(t\right) =& -\frac{1}{2g} \ln\left[e^{-2gx_0} + 2g^2 t^2\right], \end{aligned} \tag{69}$$

where k_1, k_2 and x_0 are some constants. From the above equation, v_x is obtained as follows (in $c = 1$ unit)

$$v_x\left(t\right) = -\frac{2gt}{e^{-2gx_0} + 2g^2 t^2} \tag{70}$$

In this space-time, the motion of classical particle is such that the particle moves from $x = x_0$ toward $x = 0$ with increasing speed from $t = 0$ to $t_* = e^{-2gx_0}/(\sqrt{2}g)$, and then its speed decreases from $v_* = -(\sqrt{2}/2)e^{2gx_0}$ to zero at $t \to \infty$.

4 Conclusion

In this paper, we consider the metrics which specified by a one-variable function $u(x)$, specifically when $u(x) = e^{\pm gx}$, and study the spin-0 and spin-1/2 particles in a semi-infinite laboratory in the presence of these metrics.

First we consider the conformal metric $ds^2 = u^2(x)\eta_{\mu\nu}dx^\mu dx^\nu$. For $u(x) = e^{gx}$, the Klein-Gordon and Dirac eigenfunctions and the ground-state energy of spin-0 particles are obtained exactly. The numerical values of discrete energy eigenvalues can be calculated by using these eigenfunctions. For $u(x) = e^{-gx}$, the exact eigenfunctions are obtained and it is shown that the energy levels have continuous spectrum.

Second we study the metric $ds^2 = -dt^2 + u^2(x)\eta_{ij}dx^i dx^j$. The problem has no solution for $u(x) = e^{gx}$. In the case $u(x) = e^{-gx}$, the solutions depend on the transverse momentum p. In all cases, we obtain the exact eigenfunctions. The quantized values of $E_{\rm KG}$ for $p > \sqrt{3}g/2$ and $E_{\rm D}$ for $p > g/2$ can be obtained numerically.

In all the above mentioned cases, there are some detectable differences between Fermions and Bosons, although there are some analogy between them. One of the interesting observation is that whenever the classical trajectory of particles crosses $x = 0$, that is the floor of the lab, then the quantum quantization appears in the spectrum. There is, apparently, a correspondence between the quantum-quantization and the falling property of the particles. In the cases $u = e^{gx}$ of conformal metric and $u = e^{-gx}$ of the second metric, where the

quantization appears, the classical trajectories are toward the $(-x)$-direction, while in $u = e^{-gx}$ case of the conformal metric, this is not the case.

Finally it may be interesting to bring some comments on metric (5). In this case, the Klein-Gordon wavefunction $\psi_{\rm KG}(x)$ satisfies

$$\left[-E^2 + \frac{d^2}{dx^2} + 2\frac{v'}{v}\frac{d}{dx} - \frac{u^2}{v^2}\left(p_2^2 + p_3^2\right) - m^2u^2\right]\psi_{\rm KG}(x) = 0, \tag{71}$$

and the Dirac spinor's components satisfy

$$\begin{aligned}\frac{d\phi_1}{dx} &= puv^{-1}\phi_1 - (E + mu)\,\phi_2,\\ \frac{d\phi_2}{dx} &= -puv^{-1}\phi_2 + (E - mu)\,\phi_1.\end{aligned} \tag{72}$$

Here $\psi_{\rm D} = v^{-1}u^{-1/2}\tilde{\psi}_{\rm D}$, and other notations are the same as ones introduced in Sec.2.2. By considering two different exponential functions for $u(x)$ and $v(x)$, for example $u(x) = e^{-ax}$ and $v(x) = e^{-bx}$, one may solve, not necessarily analytically, eqs.(71) and (72). The scalar-curvature R of the metric (5) for these specific u and v is $R = 6b^2e^{2ax}$. So if one chooses $a = 0$ or $b = 0$, then R becomes constant and the metric (5) reduces to metrics (2) and (1), respectively.

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